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## The Growth Rates of Ideal Coxeter Polyhedra in Hyperbolic 3-Space

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**Abstract.** In [7], Kellerhals and Perren conjectured that the growth rates of the reflection groups given by compact hyperbolic Coxeter polyhedra are always Perron numbers. We prove that this conjecture holds in the context of ideal Coxeter polyhedra in  $\mathbb{H}^3$ . Our methods allow us to bound from below the growth rates of composite ideal Coxeter polyhedra by the growth rates of its ideal Coxeter polyhedral constituents.

### 1. Introduction

An  $n$ -dimensional convex polyhedron in Euclidean, spherical or hyperbolic  $n$ -space is called a *Coxeter polyhedron* if all its dihedral angles are submultiples of  $\pi$ .

A Coxeter polyhedron is a fundamental domain of the discrete group  $\Gamma$  generated by the set  $S$  of all reflections with respect to its faces. We call  $(\Gamma, S)$  an  *$n$ -dimensional hyperbolic Coxeter group* if  $(\Gamma, S)$  is obtained from a finite volume Coxeter polyhedron in hyperbolic space  $\mathbb{H}^n$ .

Denote by  $l_S(\gamma)$  the *word length* of an element  $\gamma \in \Gamma$  with respect to  $S$ , i.e. the smallest integer  $l \geq 1$  for which there exist  $s_1, s_2, \dots, s_l \in S$  such that  $\gamma = s_1 s_2 \cdots s_l$ , and put  $l_S(1) = 0$ . The *growth series*  $f_S(t)$  of  $(\Gamma, S)$  is the formal power series  $\sum_{k=0}^{\infty} a_k t^k$ , where  $a_k$  is the number of elements  $\gamma \in \Gamma$  satisfying  $l_S(\gamma) = k$ . It is known that  $f_S(t)$  is the series expansion of a rational function whose radius of convergence  $R$  satisfies  $R < 1$ , cf. [5, 7]. The *growth rate*  $\tau$  of  $(\Gamma, S)$  (and of  $P$ ) is defined by  $\tau = \limsup_{k \rightarrow \infty} \sqrt[k]{a_k}$  and equals  $1/R$ .

In lower dimensions, there are several results about growth rates of hyperbolic Coxeter groups. In the *compact* case, Cannon–Wagreich [1] and Parry [14] showed that the growth rates of two- and three-dimensional hyperbolic Coxeter groups are Salem numbers, where a real algebraic integer  $\alpha > 1$  is called a *Salem number* if  $\alpha^{-1}$  is an algebraic conjugate of  $\alpha$  and all other algebraic conjugates of  $\alpha$  lie on the unit circle. In [7], Kellerhals and Perren proved that the growth rates of four-dimensional hyperbolic Coxeter groups with at most 6 generators are Perron numbers. Here, a real algebraic integer  $\beta > 1$  is called a *Perron number* if all its conjugates different from  $\beta$  are of absolute value strictly smaller than  $|\beta|$ .

In the *non-compact* case, Floyd proved that the growth rates of two-dimensional finite area hyperbolic Coxeter groups are Pisot numbers, where a real algebraic integer  $\gamma > 1$  is called a *Pisot number* if all its algebraic conjugates different from  $\gamma$  are of absolute value smaller than or equal to 1. Notice that Salem numbers and Pisot numbers are Perron numbers. In [10, 17], Komori and Umemoto proved that the growth rates of three-dimensional finite volume hyperbolic Coxeter groups with four or five generators are Perron numbers.

In the compact case and for each  $n \geq 2$ , Kellerhals and Perren conjectured that the growth rates of  $n$ -dimensional hyperbolic Coxeter groups are always Perron numbers. As follows from the discussion above, this conjecture is true for  $n = 2$  and  $n = 3$ . It also holds in the non-compact case, for  $n = 2$ , and for  $n = 3$  if  $|S| \leq 5$ .

In this paper, we prove that the growth rates of three-dimensional hyperbolic Coxeter groups  $(\Gamma, S)$  given by ideal Coxeter polyhedra  $P$  of finite volume are Perron numbers. Here, a Coxeter polyhedron is called *ideal* if all of its vertices are ideal points of  $\mathbb{H}^3$ , that is, they lie on the boundary at infinity  $\partial\mathbb{H}^3$ . Our main idea is to analyse the cusp properties of  $P$  and to represent the growth function  $f_{\langle P \rangle}(t) := f_S(t)$  in terms of the combinatorial-metrical data of  $P$  (see Sections 2 and 3). The reciprocal polynomial of a multiple  $g_{\langle P \rangle}(t)$  of the denominator of  $f_{\langle P \rangle}(t)$ , with  $g_{\langle P \rangle}(1) \neq 0$  then turns out to be the minimal polynomial of a Perron number, which equals the growth rate of  $P$  (see Theorem 1 in Section 4). Inspired by the work of Kellerhals [5], we prove in Section 5 that the Coxeter tetrahedron  $[(3, 6)^{[2]}]$  with a cyclic Coxeter diagram of alternating weights 3 and 6 has minimal growth rate among all ideal Coxeter polyhedra in  $\mathbb{H}^3$ . In Section 6, we bound from below the growth rate of an ideal Coxeter polyhedron  $P *_F P'$ , being a composite of two ideal Coxeter polyhedra  $P$  and  $P'$  along a face isometric to  $F$ , by the growth rates of  $P$  and  $P'$  (see Theorem 3). We illustrate our results by providing a few examples.

The main idea and results of this work are contained in Nonaka's pre-print [13], with the difference that some proofs presented here are shortened. Komori-Yukita [11, 12] proved our Theorem 1 and 2 by a partially different approach independently.

## 2. Ideal Coxeter polyhedra

Consider an ideal Coxeter polyhedron  $P$  of finite volume in  $\mathbb{H}^3$ . A vertex  $v$  of  $P$  will be called a *cusp*. The following lemma is well-known. Since it plays an important role in the proof of our main results, we provide a short proof.

LEMMA. *A cusp  $v$  of an ideal Coxeter polyhedron of finite volume in  $\mathbb{H}^3$  satisfies one of the following conditions:*

- (c<sub>1</sub>)  *$v$  belongs to three faces, and the dihedral angles at these faces are equal to  $\frac{\pi}{3}$ ,*
- (c<sub>2</sub>)  *$v$  belongs to three faces, and one of the dihedral angles at these faces is  $\frac{\pi}{2}$  and the other two dihedral angles are  $\frac{\pi}{4}$ ,*
- (c<sub>3</sub>)  *$v$  belongs to three faces, and the dihedral angles at these faces are  $\frac{\pi}{2}$ ,  $\frac{\pi}{3}$  and  $\frac{\pi}{6}$ ,*
- (c<sub>4</sub>)  *$v$  belongs to four faces, and the dihedral angles at these faces are equal to  $\frac{\pi}{2}$ .*

PROOF. Consider an ideal Coxeter polyhedron  $P$  of finite volume in the upper half-space  $\mathbb{E}^3$  equipped with the hyperbolic metric  $ds^2 = (dx_1^2 + dx_2^2 + dx_3^2)/x_3^2$  and with boundary  $\mathbb{E}^2 \cup \{\infty\}$ . Suppose, without loss of generality, that one cusp  $v$  of  $P$  is  $\infty$  and that  $v$  is the intersection of  $k \geq 3$  bounding hyperplanes of  $P$ . In this way, the dihedral angles of  $P$  at these faces, denoted by  $\frac{\pi}{n_1}, \frac{\pi}{n_2}, \dots, \frac{\pi}{n_k}$ , can be seen in a sufficiently small horosphere centred at  $\infty$ , and they satisfy the Euclidean angular condition

$$\frac{1}{n_1} + \dots + \frac{1}{n_k} = k - 2. \quad (1)$$

Assume that  $2 \leq n_1 \leq n_2 \leq \dots \leq n_k$ . Then, by (1), we deduce that

$$\frac{k}{n_1} \geq k - 2,$$

so that

$$k \leq \frac{2n_1}{n_1 - 1} = 2 + \frac{2}{n_1 - 1} \leq 2 + \frac{2}{2 - 1} = 4.$$

This means that  $k$  is equal to 3 or 4. If  $k = 4$ , then  $n_1 = n_2 = n_3 = n_4 = 2$  which corresponds to condition  $(c_4)$ . If  $k = 3$ , then  $(n_1, n_2, n_3) = (3, 3, 3), (2, 4, 4)$  or  $(2, 3, 6)$  which correspond to conditions  $(c_1), (c_2)$  and  $(c_3)$  respectively.  $\square$

In the sequel, we call a cusp  $v$  a *cusp of type*  $(c_i)$  if  $v$  satisfies condition  $(c_i)$  for some  $i$ ,  $1 \leq i \leq 4$ , according to the Lemma. We denote by  $c_i$  the number of cusps of type  $(c_i)$  for  $i = 1, \dots, 4$ . Any edge  $e$  of a polyhedron in  $\mathbb{H}^3$  belongs to exactly two faces. If the dihedral angle at  $e$  formed by these two faces is  $\frac{\pi}{n}$ , we call the edge  $e$  a  $\frac{\pi}{n}$ -edge. Let  $e_n$  be the number of  $\frac{\pi}{n}$ -edges of an ideal Coxeter polyhedron  $P$  of finite volume in  $\mathbb{H}^3$ . Denote by  $c$ ,  $e$  and  $f$  the total number of cusps, edges and faces of  $P$ , respectively. Then we obtain the following combinatorial identities.

$$c - e + f = 2, \quad (2)$$

$$3c_1 + 3c_2 + 3c_3 + 4c_4 = 2e, \quad (3)$$

$$2e_2 = c_2 + c_3 + 4c_4, \quad (4)$$

$$2e_3 = 3c_1 + c_3, \quad (5)$$

$$e_4 = c_2, \quad (6)$$

$$2e_6 = c_3. \quad (7)$$

The first identity is Euler's identity while the other identities follow by counting the number of edges passing through a cusp. For example, the identity (3) is obtained as follows. On

one hand, a cusp of type  $(c_4)$  belongs to four edges and the cusps of types different from  $(c_4)$  belong to three edges. On the other hand, any edge is adjacent to two cusps. Observe that the quantities  $c_1$ ,  $c_2$  and  $c_3$  are *even* numbers due to the identities (5) and (7).

By the Lemma, we obtain the following obvious identities.

$$c = c_1 + c_2 + c_3 + c_4, \quad (8)$$

$$c = c_2 + c_3 + c_4 + c_6. \quad (9)$$

By using the identities (2), (3) and (8), we obtain

$$c_4 = 2f - c - 4. \quad (10)$$

By substituting (10) in (8), we get

$$c_1 + c_2 + c_3 = 2(c - f + 2). \quad (11)$$

REMARK 1. If an ideal Coxeter polyhedron in  $\mathbb{H}^3$  is *right-angled*, that is, all dihedral angles are equal to  $\frac{\pi}{2}$ , then the type of each cusp is  $(c_4)$ . Hence,  $c = c_4$  and  $c = f - 2$  by (10). Furthermore, since any face has at least three cusps while each cusp is shared by at most four faces, we also obtain  $4c \geq 3f$ . Therefore,  $f \geq 8$ . As a consequence, a right-angled ideal polyhedron of minimal face number in  $\mathbb{H}^3$  must be an octahedron (see also [8, Proposition 5]).

### 3. The growth function of an ideal hyperbolic Coxeter polyhedron

Consider an ideal Coxeter polyhedron  $P \subset \mathbb{H}^3$  of finite volume and its associated reflection group  $(\Gamma, S)$  (cf. Section 1). The *growth function of  $P$* , denoted by  $f_{(P)}(t)$ , is given by the growth function  $f_S(t)$  of  $(\Gamma, S)$ . In order to study and calculate growth rates, we use the following two well-known results, due to Solomon [15], for the first instance, and to Steinberg [16], for the second one.

THEOREM. *The growth function  $f_S(t)$  of an irreducible spherical Coxeter group  $(\Gamma, S)$  is given by  $f_S(t) = \prod_{i=1}^l [m_i + 1]$  where  $[n] := 1 + t + \dots + t^{n-1}$  and  $\{m_1 = 1, m_2, \dots, m_l\}$  is the set of exponents (as defined in Section 3.16 of [2]) of  $(\Gamma, S)$ .*

THEOREM. *Let  $(\Gamma, S)$  be a Coxeter group. Denote by  $(\Gamma_T, T)$  the Coxeter subgroup of  $(\Gamma, S)$  generated by the subset  $T \subseteq S$ , and let  $f_T(t)$  be its growth function. Put  $\mathcal{F} = \{T \subseteq S \mid \Gamma_T \text{ is finite}\}$ . Then*

$$\frac{1}{f_S(t^{-1})} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(t)}.$$

In [2, 7], for example, one can find the list of the exponents and the growth polynomials of all irreducible spherical (finite) Coxeter groups, classically denoted by  $A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_2(m)$ , in terms of  $[n] = 1 + t + \dots + t^{n-1}$ ,  $[n, m] := [n][m]$  and so on.

From now on, we focus on ideal Coxeter polyhedra  $P$  of finite volume in  $\mathbb{H}^3$  and their growth functions  $f_{\langle P \rangle}(t)$  as given by the growth functions  $f_S(t)$  of the associated reflection groups  $(\Gamma, S)$ . Since  $P$  has no vertices in  $\mathbb{H}^3$ , only faces and edges of  $P$  can be stabilised by *finite* subgroups of  $(\Gamma, S)$ . In particular, a non-trivial finite Coxeter subgroup  $(\Gamma_T, T)$  of  $(\Gamma, S)$  satisfies  $|T| = 1$  or  $|T| = 2$ . For  $|T| = 1$ ,  $(\Gamma_T, T)$  is isomorphic to  $A_1$  with growth function  $[2] = 1 + t$ , and the number of these subgroups in  $(\Gamma, S)$  equals  $\mathfrak{f}$ . For  $|T| = 2$ ,  $(\Gamma_T, T)$  is isomorphic to the dihedral group  $I_2(m)$  of order  $2m$ ,  $m \geq 2$ , and with growth function  $[2, m]$ . Observe that the group  $I_2(2)$  is isomorphic to  $A_1 \times A_1$  with growth function  $[2, 2] = (1 + t)^2$ .

By means of Steinberg's Theorem, we can write

$$\begin{aligned} \frac{1}{f_{\langle P \rangle}(t^{-1})} &= 1 - \frac{\mathfrak{f}}{[2]} + \frac{\mathfrak{e}_2}{[2, 2]} + \frac{\mathfrak{e}_3}{[2, 3]} + \frac{\mathfrak{e}_4}{[2, 4]} + \frac{\mathfrak{e}_6}{[2, 6]} \\ &= 1 - \frac{\mathfrak{f}}{1+t} + \frac{6\mathfrak{f} - 2\mathfrak{c} - \mathfrak{c}_1 - 12}{2(1+t)^2} + \frac{2\mathfrak{c} - 2\mathfrak{f} + 2\mathfrak{c}_1 - \mathfrak{c}_2 + 4}{2(1+t)(1+t+t^2)} \\ &\quad + \frac{\mathfrak{c}_2}{(1+t)^2(1+t^2)} + \frac{2\mathfrak{c} - 2\mathfrak{f} - \mathfrak{c}_1 - \mathfrak{c}_2 + 4}{2(1+t)^2(1+t+t^2)(1-t+t^2)}. \end{aligned}$$

The second equality is obtained by using the identities (4)–(7), (10) and (11). By simplifying the expression above, we get the following combinatorial formula for the growth function.

**PROPOSITION.** *Let  $P \subset \mathbb{H}^3$  be an ideal Coxeter polyhedron of finite volume with  $\mathfrak{c}$  cusps and  $\mathfrak{f}$  faces. Then, its growth function  $f_{\langle P \rangle}(t)$  is given by*

$$f_{\langle P \rangle}(t) = \frac{[2, 2, 3](1+t^2)(1-t+t^2)}{(t-1)g_{\langle P \rangle}(t)}, \quad (12)$$

wherein  $g_{\langle P \rangle}(t)$  is a polynomial of degree 7 with combinatorial coefficients given by

$$\begin{aligned} g_{\langle P \rangle}(t) &= (\mathfrak{c} - 1)t^7 + (\mathfrak{c} - \mathfrak{f} + 1)t^6 + (\mathfrak{c} + \mathfrak{f} - \mathfrak{c}_1/2 - 4)t^5 + \\ &\quad + (2\mathfrak{c} - 2\mathfrak{f} + (\mathfrak{c}_1 - \mathfrak{c}_2)/2 + 2)t^4 + (2\mathfrak{f} - (\mathfrak{c}_1 - \mathfrak{c}_2)/2 - 6)t^3 + \\ &\quad + (\mathfrak{c} - \mathfrak{f} + \mathfrak{c}_1/2)t^2 + (\mathfrak{f} - 3)t - 1. \end{aligned} \quad (13)$$

Since the radius of convergence  $R < 1$  of the growth series  $f_{\langle P \rangle}(t) = \sum_{k=0}^{\infty} a_k t^k$  is the smallest real zero of  $g_{\langle P \rangle}(t)$  in the interval  $]0, 1[$  whose reciprocal  $\tau = 1/R$  is the growth rate of  $P$ , it suffices to analyse the zero distribution of  $g_{\langle P \rangle}(t)$  in the unit disc of  $\mathbb{C}$ .

**EXAMPLE 1.** Consider one of the three ideal Coxeter tetrahedra in  $\mathbb{H}^3$  denoted by  $T_1 = [(3, 6)^{[2]}]$ ,  $T_2 = [4^{[4]}]$  and  $T_3 = [3^{[3, 3]}]$  (cf. [4, p. 348]). In Figure 1 we provide a description of their metrical structure where the weight  $k \in \{2, 3, 4, 6\}$  indicates the presence of a  $\frac{\pi}{k}$ -edge. Each tetrahedron satisfies  $\mathfrak{c} = \mathfrak{f} = 4$  and  $\mathfrak{e} = 6$ . More specifically, all the cusps of  $T_1$ ,  $T_2$  and  $T_3$ , respectively, are only of one type, namely of type  $(c_3)$ ,  $(c_2)$  and  $(c_1)$ , respectively. Observe that  $T_3$  is an ideal regular tetrahedron of dihedral angle  $\frac{\pi}{3}$ . By (13) of

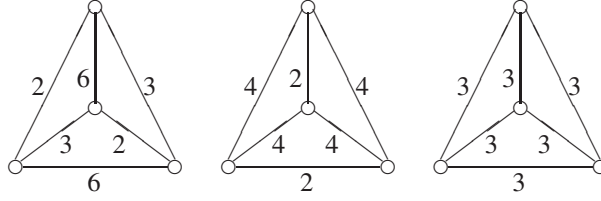


FIGURE 1. Metrical structure of the ideal Coxeter tetrahedra  $T_1, T_2, T_3 \subset \mathbb{H}^3$

the Proposition, we obtain the following expressions for the denominator polynomial  $g_i(t)$  of the growth function  $f_i(t)$  of  $T_i$  for  $i = 1, 2, 3$  (compare with [10, Proposition 1]).

$$g_1(t) = 3t^7 + t^6 + 4t^5 + 2t^4 + 2t^3 + t - 1 = (t^2 + 1)(3t^5 + t^4 + t^3 + t^2 + t - 1), \quad (14)$$

$$g_2(t) = 3t^7 + t^6 + 4t^5 + 4t^3 + t - 1 = [3](t^2 - t + 1)(3t^3 + t^2 + t - 1), \quad (15)$$

$$g_3(t) = 3t^7 + t^6 + 2t^5 + 4t^4 + 2t^2 + t - 1 = [2](t^2 + 1)(t^2 - t + 1)(3t^2 + t - 1). \quad (16)$$

For the growth rate  $\tau_1$  of  $T_1$ , we get the numerical estimate  $\tau_1 \sim 2.03074$  by exploiting (14). By (15), the reciprocal value of the (single) real root of cubic polynomial in  $g_2(t)$  (being of discriminant  $-304$ ) gives the growth rate  $\tau_2$  of  $T_2$  and equals

$$\tau_2 = \frac{1}{3}(\sqrt[3]{2(23 - 3\sqrt{57})} + \sqrt[3]{2(23 + 3\sqrt{57})}) \sim 2.13040.$$

By (16), we see that the growth rate  $\tau_3$  of  $T_3$  equals  $(1 + \sqrt{13})/2 \sim 2.30278$ .

By comparison, we obtain  $\tau_1 < \tau_2 < \tau_3$ .

**EXAMPLE 2.** Consider a right-angled ideal Coxeter polyhedron  $P \subset \mathbb{H}^3$ . Then, all cusps are of type  $(c_4)$  so that  $\mathfrak{c} = c_4 \geq 5$ ,  $\mathfrak{e} = 2\mathfrak{c}$  and  $\mathfrak{f} = \mathfrak{c} + 2$  by the identities (2) and (3). In this case, equation (13) yields

$$\begin{aligned} g_{(P)}(t) &= (\mathfrak{c} - 1)t^7 - t^6 + 2(\mathfrak{c} - 1)t^5 - 2t^4 + 2(\mathfrak{c} - 1)t^3 - 2t^2 + (\mathfrak{c} - 1)t - 1 \\ &= \{(\mathfrak{c} - 1)t - 1\}(t^6 + 2t^4 + 2t^2 + 1) \\ &= \{(\mathfrak{c} - 1)t - 1\}[3](1 + t^2)(1 - t + t^2). \end{aligned}$$

By (12), we obtain the expression

$$f_{(P)}(t) = \frac{(t + 1)^2}{(t - 1)\{(\mathfrak{c} - 1)t - 1\}}, \quad (17)$$

which shows that the growth rate of  $P$  is the Pisot number (and Perron number)  $\tau = \mathfrak{c} - 1$ .

In the special case of a right-angled ideal octahedron  $O$  satisfying  $\mathfrak{c} = 6$ , the result (17) gives  $\tau(O) = 5$ .

#### 4. Perron numbers and growth rates of ideal Coxeter polyhedra

The aim of this section is to prove our main theorem.

**THEOREM 1.** *The growth rate of an ideal Coxeter polyhedron of finite volume in  $\mathbb{H}^3$  is a Perron number.*

**PROOF.** Let  $P \subset \mathbb{H}^3$  be an ideal Coxeter polyhedron of finite volume with  $c$  cusps,  $e$  edges and  $f$  faces satisfying  $c - e + f = 2$ . By Example 2, we know that the growth rate of a right-angled polyhedron is a Perron number. Therefore, we may assume in the sequel that  $P$  is *not* right-angled, that is, the number  $c - c_4 = c_1 + c_2 + c_3$  is positive (and even). We perform the proof in two steps.

Firstly, and as a warm-up, we assume that  $P$  is a *simple* polyhedron, that is, each cusp lies on exactly three faces. It follows that  $2e = 3c$  and  $c = 2(f - 2)$ . For the growth function  $f_{\langle P \rangle}(t)$  of  $P$ , equations (12) and (13) of the Proposition yield

$$f_{\langle P \rangle}(t) = \frac{[2, 2, 3](1 + t^2)(1 - t + t^2)}{(t - 1)g_{\langle P \rangle}(t)}$$

with

$$\begin{aligned} g_{\langle P \rangle}(t) &= (c - 1)t^7 + (f - 3)t^6 + (c_1/2 + c_2 + c_3 + c_4 + f - 4)t^5 + \\ &\quad + (2(f - 3) + (c_1 - c_2)/2)t^4 + (c_1/2 + 3c_2/2 + c_3 + c_4 - 2)t^3 + \\ &\quad + (f + c_1/2 - 4)t^2 + (f - 3)t - 1 =: \sum_{k=1}^7 n_k t^k - 1 \in \mathbb{Z}[t]. \end{aligned} \quad (18)$$

By a result of Komori-Umemoto [10, Lemma 1], it suffices to prove that all coefficients  $n_k$ ,  $k \geq 1$ , of the polynomial  $g_{\langle P \rangle}(t)$  in (18) are non-negative and that the greatest common divisor of  $\{k \in \mathbb{N} \mid n_k \neq 0\}$  is 1. In fact, their result will then guarantee the existence of a real number  $r_0 \in ]0, 1[$  which is the unique zero of  $g_{\langle P \rangle}(t)$  having the smallest absolute value of all zeros of  $g_{\langle P \rangle}(t)$ . This  $r_0$  will be equal to  $R = 1/\tau$  so that  $\tau$  is a Perron number.

By (18), and since  $c, f \geq 4$ , it is obvious that the coefficients  $n_k$  for  $k \neq 4$  are non-negative. As for the coefficient  $n_4 = 2(f - 3) + (c_1 - c_2)/2$ , we use  $c = 2(f - 2)$  and rewrite  $n_4 = 3c_1/2 + c_2/2 + c_3 + c_4 - 2$  and since  $c = c_1 + c_2 + c_3 + c_4 \geq 4$ , we have  $n_4 \geq 0$ . Since the coefficients  $n_7 = c - 1 = 2f - 5$  and  $n_6 = f - 3$  are positive with coprime indices, the conclusion follows in the case when  $P$  is simple.

Secondly, assume that  $P$  is *not* simple, that is,  $P$  has at least one cusp of type  $(c_4)$  so that  $c > c_4 \geq 1$ . By (2), (3) and (10), we derive that  $f = (c_1 + c_2 + c_3)/2 + c_4 + 2$ . Hence, by (13) and as above, we get

$$\begin{aligned} g_{\langle P \rangle}(t) &= (c - 1)t^7 + \left(\frac{c_1 + c_2 + c_3}{2} - 1\right)t^6 + \left(\frac{2c_1 + 3c_2 + 3c_3 + 4c_4}{2} - 2\right)t^5 + \\ &\quad + \left(\frac{3c_1 + c_2 + 2c_3}{2} - 2\right)t^4 + \left(\frac{c_1 + 3c_2 + 2c_3 + 2c_4}{2} - 2\right)t^3 + \end{aligned}$$

$$+ \left( \frac{2c_1 + c_2 + c_3}{2} - 2 \right) t^2 + (f - 3)t - 1 =: \sum_{k=1}^7 n_k t^k - 1 \in \mathbb{Z}[t]. \quad (19)$$

Since  $P$  is not right-angled, with  $f \geq 5$  and with  $c_1 + c_2 + c_3 > 0$  an even number, the identity (19) allows to conclude that all polynomial coefficients  $n_k$  are non-negative with strictly positive coefficients  $n_7$  and  $n_5$  having coprime indices. Therefore, the growth rate  $\tau$  of  $P$  is a Perron number.  $\square$

## 5. The minimal growth rate of ideal Coxeter polyhedra

In this section, we determine the polyhedron of minimal growth rate among all ideal Coxeter polyhedra in  $\mathbb{H}^3$  of finite volume. Our approach is inspired by the work of Kellerhals [5] wherein she determined the (unique) non-compact finite volume Coxeter polyhedron of minimal growth rate. She showed that the Coxeter tetrahedron  $[3, 3, 6]$  with only one cusp (being of type  $(c_3)$ ) has minimal growth rate  $\tau_0 \sim 1.29647$ , and that  $\tau_0$  is not a Pisot number but a Perron number with minimal polynomial  $t^7 - t^3 - t^2 - t - 1$  (cf. [5, p. 1013]). It follows from this result that the minimal growth rate of *ideal* Coxeter polyhedra of finite volume must be strictly larger than  $\tau_0$ .

As hinted by [5], the three ideal Coxeter tetrahedra  $T_1, T_2, T_3$  with growth rates  $\tau_1 < \tau_2 < \tau_3 \sim 2.30270$  in Example 1 are of special interest and allow us to prove the following result.

**THEOREM 2.** *The Coxeter tetrahedron  $T_1 = [(3, 6)^{[2]}]$  has minimal growth rate among all ideal polyhedra of finite volume in  $\mathbb{H}^3$ , and as such is unique. Its growth rate  $\tau_1 \sim 2.03074$  is the Perron number with minimal polynomial  $t^5 - t^4 - t^3 - t^2 - t - 3$ .*

**PROOF.** By Example 1, it suffices to prove that the growth rate of any ideal finite volume Coxeter polyhedron  $P$  which is *not* a tetrahedron is larger than  $\tau_3$ , the growth rate of the tetrahedron  $T_3$ . Since  $P$  is not a tetrahedron, we have  $c, f \geq 5$ . We will show that  $1/f_{\langle T_3 \rangle}(t) > 1/f_{\langle P \rangle}(t)$  for all  $t \in ]0, 1[$ . By our Proposition, it is enough to prove that  $g_{\langle P \rangle}(t) > g_{\langle T_3 \rangle}(t) = g_3(t)$  for all  $t \in ]0, 1[$ . Thus, consider the difference of the functions given by (13) and (16)

$$g_{\langle P \rangle}(t) = (c-1)t^7 + (c-f+1)t^6 + (c+f-c_1/2-4)t^5 + (2c-2f+(c_1-c_2)/2+2)t^4 \\ + (2f-(c_1-c_2)/2-6)t^3 + (c-f+c_1/2)t^2 + (f-3)t - 1,$$

$$g_3(t) = 3t^7 + t^6 + 2t^5 + 4t^4 + 2t^2 + t - 1 = [2](t^2+1)(t^2-t+1)(3t^2+t-1).$$

An easy computation and the bounds  $c, f \geq 5$  yield

$$g_{\langle P \rangle}(t) - g_3(t) = t^2(1+2t^2+t^3+t^4+t^5)(c-4) + t(1-t)(1+t^2)^2(f-4) \\ + t^2(1-t)(1+t^2)\left(\frac{c_1}{2}-2\right) + t^3(1-t)\frac{c_2}{2}$$



$$\begin{aligned}
&\geq t^2(1 + t^2 + t^3 + t^4 + t^5) + t(1 - t)(1 + t^2)^2 - 2t^2(1 - t)(1 + t^2) \\
&= t(1 - 2t + 4t^2 - 3t^3 + 4t^4 + t^6) \\
&= t\{(1 - t^2) + 3t^2(1 - t) + t^4(4 + t^2)\} > 0
\end{aligned}$$

for all  $t \in ]0, 1[$ . Therefore, the growth rate  $\tau$  of any ideal Coxeter polyhedron of finite volume which is not a tetrahedron is strictly larger than  $\tau_3$ .  $\square$

REMARK 2. For the growth rates  $\tau_i$  of the three ideal Coxeter tetrahedra  $T_i$ ,  $i = 1, 2, 3$ , we have that  $\tau_1 < \tau_2 < \tau_3 \sim 2.30278$  by Example 1. A similar arrangement holds for their volumes  $v_i = \text{vol}_3(T_i)$ ,  $i = 1, 2, 3$ . These values are known and satisfy (cf. [4, p. 348], for example)

$$\begin{aligned}
v_1 &= \text{vol}_3(T_1) = \frac{5}{2} \text{JI}\left(\frac{\pi}{3}\right) \sim 0.84579, \\
v_2 &= \text{vol}_3(T_2) = 2 \text{JI}\left(\frac{\pi}{4}\right) \sim 0.91597, \\
v_3 &= \text{vol}_3(T_3) = 3 \text{JI}\left(\frac{\pi}{3}\right) \sim 1.01492,
\end{aligned}$$

where

$$\text{JI}(x) = \frac{1}{2} \sum_{r=1}^{\infty} \frac{\sin(2rx)}{r^2} = - \int_0^x \log |2 \sin t| dt, \quad x \in \mathbb{R},$$

is Lobachevsky's function. In particular, we get the inequalities  $v_1 < v_2 < v_3$ .

## 6. Bounding growth rates of composite ideal Coxeter polyhedra

As a motivation for this section, consider an ideal Coxeter pyramid  $P \subset \mathbb{H}^3$  of finite volume, that is,  $P$  is the convex cone with apex on  $\partial\mathbb{H}^3$  and base which is an ideal  $k$ -polygon. By the Lemma, any of the cusps of  $P$  belongs to three or four faces of  $P$ , that is,  $P$  is either an ideal Coxeter tetrahedron  $T_i$  for some  $1 \leq i \leq 3$  or a cone over an ideal rectangle. In the latter case, the Lemma also implies that there are only two ideal Coxeter square pyramids  $P_1$  and  $P_2$ , say. The pyramid  $P_1$  has a rectangular face with two opposite  $\frac{\pi}{3}$ -edges and two opposite  $\frac{\pi}{6}$ -edges, while the pyramid  $P_2$  has a rectangular face with only  $\frac{\pi}{4}$ -edges. Their metrical structure is depicted in Figure 2.

Now, both pyramids  $P_1$  and  $P_2$  can be dissected into ideal Coxeter polyhedra which are, moreover, tetrahedra of small growth rates. More concretely, consider the hyperplane whose closure passes through the apex of  $P_1$  and two opposite cusps belonging to the rectangular face of  $P_1$ . Cutting  $P_1$  by this hyperplane, we obtain two tetrahedra isometric to  $T_1$ . In a similar way, the pyramid  $P_2$  admits the decomposition into two isometric copies of  $T_2$ . Therefore, by Remark 2,

$$\text{vol}(P_1) = 2\text{vol}(T_1) = 5 \text{JI}\left(\frac{\pi}{3}\right), \quad \text{vol}(P_2) = 2\text{vol}(T_2) = 4 \text{JI}\left(\frac{\pi}{4}\right).$$

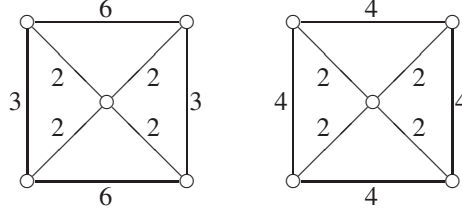


FIGURE 2. Metrical structure of the ideal Coxeter pyramids  $P_1, P_2 \subset \mathbb{H}^3$

As for the growth rates of  $P_1$  and  $P_2$ , we first determine their growth denominator polynomials by exploiting the Proposition (cf. also [17]), that is,

$$g_{\langle P_1 \rangle}(t) = 4t^5 + t^4 + 2t^3 + t^2 + 2t - 1, \quad g_{\langle P_2 \rangle}(t) = 4t^3 + t^2 + 2t - 1. \quad (20)$$

Then, we get the estimates  $\tau(P_1) \sim 2.74738$  and  $\tau(P_2) \sim 2.84547$ . It follows that

$$\tau(P_1) > \tau_1, \quad \tau(P_2) > \tau_2.$$

In fact, the following general result is true.

**THEOREM 3.** *Let  $P$  and  $P'$  be ideal Coxeter polyhedra in  $\mathbb{H}^3$  with growth rates  $\tau(P)$  and  $\tau(P')$ . Suppose that  $P$  has a face  $F$  which is isometric to a face  $F'$  of  $P'$ , and denote by  $P *_F P'$  the ideal polyhedron arising by glueing  $P$  to  $P'$  along their isometric faces  $F$  and  $F'$ . If  $P *_F P'$  is an ideal Coxeter polyhedron, then its growth rate  $\tau(P *_F P')$  is greater than  $\max\{\tau(P), \tau(P')\}$ .*

**REMARK 3.** The reflection group associated to the ideal Coxeter polyhedron  $P *_F P'$  is the free product of the reflection groups associated to the ideal Coxeter polyhedra  $P$  and  $P'$  amalgamated by the reflection group associated to the ideal polygon  $F$ , cf. Theorem 6.1 of [3].

**PROOF.** We make use of our Proposition in order to compare the growth functions  $f_{\langle P *_F P' \rangle}(t)$  and  $f_{\langle P \rangle}(t)$  of  $P *_F P'$  and  $P$ , say.

To this end, let  $c_{m,n}(F)$  be the number of cusps of  $F$  which lie on a  $\frac{\pi}{m}$ -edge and a  $\frac{\pi}{n}$ -edge of  $F$ . If  $F$  has a  $\frac{\pi}{4}$ -edge, this edge must coincide with an (isometric copy of a)  $\frac{\pi}{4}$ -edge of  $F'$  since  $P *_F P'$  is a Coxeter polyhedron. Note that the dihedral angle of  $P *_F P'$  at this edge is  $\frac{\pi}{2}$ . Thus, any cusp counted in  $c_{4,4}(F)$  corresponds to a cusp of  $P'$  which is counted in  $c_{4,4}(F')$ . In addition, the type of this cusp in  $P *_F P'$  is  $c_4$ , and this cusp is counted in  $c_4(P *_F P')$ .

Let  $F_1$  be a face of  $P$  which is adjacent to  $F$  at a  $\frac{\pi}{2}$ -edge, and let  $F'_1$  be a face of  $P'$  which is adjacent to  $F'$  at a  $\frac{\pi}{2}$ -edge. If these two  $\frac{\pi}{2}$ -edges are identified by the glueing, then these edges disappear in  $P *_F P'$ . In particular, the hyperplane containing  $F_1$  is - up to isometry - identical to the hyperplane containing  $F'_1$ .

From the above, we get the following conclusion. Firstly, any cusp counted in  $c_{4,4}(F)$  coincides with both a cusp counted in  $c_{4,4}(F')$  and a cusp counted in  $c_4(P *_F P')$  since  $P *_F P'$  is also Coxeter polyhedron. Secondly, any cusp counted in  $c_{2,4}(F)$  must coincide with a cusp counted in both  $c_{2,4}(F')$  and  $c_2(P *_F P')$ .

The dihedral angle of  $P *_F P'$  formed by a  $\frac{\pi}{6}$ -edge of  $F$  and a  $\frac{\pi}{6}$ -edge of  $F'$  is equal to  $\frac{\pi}{3}$ . Thus, by glueing  $F$  and  $F'$ , any cusp counted in  $c_{2,6}(F)$  coincides with a cusp counted in  $c_{2,6}(F')$ , and it is also counted in  $c_1(P *_F P')$ . Since  $\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$ , any cusp counted in  $c_{3,6}(F)$  coincides with a cusp counted in  $c_{6,3}(F')$ , and it is also counted in  $c_4(P *_F P')$ .

By the above investigation, we obtain the identities

$$c(P *_F P') = c(P) + c(P') - c(F), \quad (21)$$

$$c_1(P *_F P') = c_1(P) + c_1(P') + c_{2,6}(F), \quad (22)$$

$$c_2(P *_F P') = c_2(P) + c_2(P') - 2c_{4,4}(F) - c_{2,4}(F). \quad (23)$$

Denote by  $f$  the number of faces of a polyhedron, and let  $e_k(F)$  be the number of  $\frac{\pi}{k}$ -edges of the face  $F$ . Since any  $\frac{\pi}{2}$ -edge of  $F$  (and of  $F'$ ) disappears in  $P *_F P'$ , we deduce that

$$f(P *_F P') = f(P) + f(P') - e_2(F) - 2. \quad (24)$$

From the equations (21)–(23), it follows that  $c(P *_F P') - c(P) > 0$ ,  $c_1(P *_F P') - c_1(P) \geq 0$  and  $c_2(P *_F P') - c_2(P) \geq 0$ .

As for (24), we have that  $f(P *_F P') - f(P) = f(P') - e_2(F) - 2 \geq 0$  as well, since  $f(P') \geq e(F) + 1 \geq e_2(F) + 1$ , and  $e(F) = e_2(F)$  only for  $f(P') > e(F) + 1$  by the Lemma.

By the Proposition, we get, for all  $t \in ]0, 1[$ ,

$$\begin{aligned} g_{(P *_F P')}(t) - g_{(P)}(t) &= t^2(1+t)(1+t+t^2+t^4)(c(P *_F P') - c(P)) \\ &\quad + t(1+t^2)^2(1-t)(f(P *_F P') - f(P)) \\ &\quad + t^2(1+t^2)(1-t)(c_1(P *_F P') - c_1(P)) \\ &\quad + t^3(1-t)(c_2(P *_F P') - c_2(P)) > 0, \end{aligned}$$

and therefore,

$$(t-1)g_{(P *_F P')}(t) < (t-1)g_{(P)}(t), \quad \forall t \in ]0, 1[.$$

Hence, by the Proposition, we obtain

$$\frac{1}{f_{(P *_F P')}(t)} < \frac{1}{f_{(P)}(t)}, \quad \forall t \in ]0, 1[ ,$$

so that  $\tau(P *_F P') > \max\{\tau(P), \tau(P')\}$ .  $\square$

**EXAMPLE 3.** Consider an ideal right-angled octahedron  $O$  in  $\mathbb{H}^3$  whose growth rate is the Pisot number  $\tau(O) = 5$  (cf. Example 2). It is obvious that  $O$  can be decomposed into two isometric copies of the Coxeter pyramid  $P_2$ . By (20),  $P_2$  has the growth denominator

polynomial  $g_{(P_2)}(t) = 4t^3 + t^2 + 2t - 1$  and growth rate  $\tau(P_2) \sim 2.84547$  which is smaller than 5.

EXAMPLE 4. Consider an ideal regular cube  $C \subset \mathbb{H}^3$  of dihedral angle  $\frac{\pi}{3}$ . By the Proposition, its growth function is given by

$$f_{(C)}(t) = \frac{[2, 3]}{(t-1)(7t^2 + 3t - 1)}$$

providing the growth rate  $\tau(C) = \frac{42+3\sqrt{37}}{28} \sim 4.54138$ . The cube  $C$  admits a symmetry plane  $H$  so that  $C = P *_F P$  where  $P$  is an ideal triangular Coxeter prism and  $F = C \cap H$ . The dihedral angles of  $P$  at edges not belonging to  $F$  are equal to  $\frac{\pi}{3}$ , while the quadrilateral face  $F$  has two opposite  $\frac{\pi}{2}$ -edges and two opposite  $\frac{\pi}{6}$ -edges. For the growth function of  $P$ , this implies that

$$f_{(P)}(t) = \frac{[2, 2, 3](t^2 - t + 1)}{(t-1)(5t^5 + 2t^4 + t^3 + 3t^2 + 2t - 1)}.$$

It follows that the growth rate estimate  $\tau(P) \sim 3.16204$  satisfies  $\tau(P) < \tau(C)$ .

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